Resonator Theory

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First develop the general framework. Consider the differential equation
$\ddot{x} + 2\alpha\dot{x} + {\omega_0}^2 x = y = y_0 e^{j\omega t} = \frac{y_1}{y_2} e^{j\omega t}$
where x, y are function of t and $\alpha, \omega_0, y_0, y_1, y_2$ are real constants and $\alpha, \omega_0 > 0$.

1 Homogenous Solution

Since this is a constant coefficient linear differential equation, the homogenous solution is

$$x = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

where c_1, c_2 are arbitrary complex constants and s_1, s_2 are solutions of the characteristic equation

$$s^2 + 2\alpha s + {\omega_0}^2 = 0$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - {\omega_0}^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - {\omega_0}^2}$$

1.1 Case 1: Overdamped

$$\alpha > \omega_0$$

Which case is best for pulse, ring-down detection?

1.2 Case 2: Critically Damped

$$\alpha = \omega_0$$

1.3 Case 3: Underdamped, Oscillatory

$$\alpha < \omega_0$$

In this case, the discriminant is negative so

$$s_1 = -\alpha + j\sqrt{{\omega_0}^2 - \alpha^2}$$

$$s_2 = -\alpha - j\sqrt{{\omega_0}^2 - \alpha^2}$$

Let $\omega_d = \sqrt{{\omega_0}^2 - \alpha^2}$

$$s_1 = -\alpha + j\omega_d$$

$$s_2 = -\alpha - j\omega_d$$

$x = c_1 e^{-\alpha t} e^{j\omega_d t} + c_2 e^{-\alpha t} e^{-j\omega_d t}$
Since $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$ and $\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$
$x = c_1 e^{-\alpha t} \cos\omega_d t + c_2 e^{-\alpha t} \sin\omega_d t$
or
$x = c e^{-\alpha t} \cos(\omega_d t + \phi)$
where c_1, c_2 have been redefined.
α is the *dampling factor.*

For small α the system decays at an approximate angular frequency of ω_0, the *undamped natural frequency.* The actual frequency of oscillation is ω_d, the *damped natural frequency.*

An important parameter is the size of ω_0 to α. So we define Q, the *Quality Factor* of the system as $Q = \omega_0/2\alpha$. From (*), it is clear that for large Q, $\omega_d \approx \omega_0$.

$$\omega_d = \omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

For $Q > 5$, the ratio $\omega_d/\omega_0 > .994$.

Also note that $e^{-\alpha t}$ can be written as $e^{-\frac{\omega_0}{2Q}t}$. That is, the damping α is inversely proportional to Q.

[energy in a RLC decays to zero in approximately Q cycles]

2 Particular Solution

Let's examine the response of the system to a sinusoidal input. Let $y = y_0 e^{j\omega t}$ where y_0 is real. One can solve this equation by annihilation or Laplace transforms, etc. but it's so simple that we will just guess the solution. Letting $x = ce^{j\omega t}$ where c is complex and plugging into (*) yields

$$c = \frac{y_0}{\omega_0^2 - \omega^2 + 2\alpha j\omega} = \frac{y_0}{\sqrt{({\omega_0}^2 - \omega^2)^2 + 4\alpha^2\omega^2}} \exp\left(-j \arctan\frac{2\alpha\omega}{{\omega_0}^2 - \omega^2}\right)$$

where the last term is c in polar coordinates.

The transfer function of the system is

$$H(jw) = \frac{x}{y} = \frac{1}{\sqrt{({\omega_0}^2 - \omega^2)^2 + 4\alpha^2\omega^2}} \exp\left(-j \arctan \frac{2\alpha\omega}{{\omega_0}^2 - \omega^2}\right) \tag{1}$$

Let

phase

$$\begin{aligned} \delta &= \arctan \frac{2\alpha\omega}{{\omega_0}^2 - \omega^2} \\ &= \arctan \frac{\frac{1}{Q}}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}} \end{aligned}$$

magnitude

$$\begin{aligned} A &= \frac{1}{\sqrt{({\omega_0}^2 - \omega^2)^2 + 4\alpha^2\omega^2}} \\ &= \frac{1}{{\omega_0}^2} \frac{\frac{\omega_0}{\omega}}{\sqrt{(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + \frac{1}{Q^2}}} \end{aligned}$$

As ω goes from 0 to infinity, the phase of the transfer function goes from 0 to $-\pi$. When $\omega = \omega_0$, the phase is exactly $-\pi/2$.

The amplitude of the transfer function is maximized at a frequency less than ω_0 (where does ω_d fit in?). For the rest of the discussion assume that $Q > 1/\sqrt{2}$ ($\omega_0 > \sqrt{2}\alpha$) implies that the magnitude of the transfer function is maximized at

$$\omega_m = \sqrt{{\omega_0}^2 - 2\alpha^2}$$

For high Q systems, $\omega_m \approx \omega_0$. This can be seen from the easily verified equation:

$$\omega_m = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}$$

.

If $Q > 5$, the ratio ω_m/ω_0 is greater than .9899.

In general $\omega_m < \omega_d < \omega_0$.

2.1 Gain

Let $H_0 = |H(0)| = 1/{\omega_0}^2$. Then it is easy to verify that $|H(j\omega_0)| = QH_0$ and

$$H(j\omega_m) = H_0 \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}$$

The ratio $|H(j\omega_m)|/H_0$ is always greater than Q but for $Q > 5$, the ratio $|H(j\omega_m)|/H_0$ is within 1.005% of the actual value of Q.

3 Power

The average power of a system with reactance is often more useful than instantaneous power of the system. Drive the system with an input $y = \frac{y_1}{y_2}e^{j\omega t}$. The coefficient in the drive is $\frac{y_1}{y_2}$ rather than just y_0 because we are going to define an abstract average power P as

$$P = \frac{1}{2}Re\{\overline{y_2 y}\frac{dx}{dt}\} = Re\{y_2 y\}Re\{\frac{dx}{dt}\}$$

The y_2 term really belongs to the system and the driving force is the $y_1 e^{j\omega t}$ term. The response of the system is

$$x = \frac{y_1}{y_2}Ae^{j(\omega t-\delta)}$$
$$\frac{dx}{dt} = \frac{y_1}{y_2}j\omega Ae^{j(\omega t-\delta)}$$

Therefore

$$P = \frac{1}{2}\frac{{y_1}^2}{y_2}\omega A\sin\delta$$

Since (I think this is correct)

$$\begin{aligned}\sin\delta &= \sin\arctan\frac{\frac{1}{Q}}{\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}} \\ &= \frac{1}{\sqrt{(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0})^2+\frac{1}{Q^2}}}\end{aligned}$$

we get

$$P = \frac{{y_1}^2}{2y_2\omega_0 Q}\frac{1}{(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0})^2+\frac{1}{Q^2}}$$

The maximum power P_m occurs at $\omega = \omega_0$

$$P_m = Q\frac{{y_1}^2}{2y_2\omega_0} = \frac{1}{\alpha}\frac{{y_1}^2}{4y_2}$$

3.1 Half Power Points

Evaluate the frequencies at which the power is $P_m/2$. We assume that we have a high Q system so that we only evaluate P for frequencies near ω_0.

Assuming $\omega \approx \omega_0$ yields

$$\begin{aligned}
\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} &= \frac{{\omega_0}^2 - \omega^2}{\omega\omega_0} \\
&= \frac{(\omega_0 - \omega)(\omega_0 - \omega)}{\omega\omega_0} \\
&\approx \frac{2\omega_0(\omega_0 - \omega)}{{\omega_0}^2} \\
&= 2\frac{\omega_0 - \omega}{\omega_0}
\end{aligned}$$

Using this approximation in the equation $P(\omega) = P_m/2$ yields

$$(\omega_0 - \omega)^2 = \alpha^2$$

Letting $\omega = \omega + \Delta\omega$ where $\Delta\omega > 0$ yields

$$\Delta\omega = \alpha$$

$$2\Delta\omega = \frac{\omega_0}{Q}$$

$$Q = \frac{\omega_0}{2\Delta\omega}$$

The bandwidth is twice the damping factor α. The bandwidth is ω_0/Q. Q is the maximum power frequency divided by the bandwidth.

4 Loading Mode

4.1 Inductive Coupling

Use diagram from Reitz, p.280

It is inappropriate to model the system with the transformer equations

$$\begin{aligned}
\frac{V_1}{V_2} &= \frac{N_p}{N_s} \\
\frac{I_1}{I_2} &= \frac{N_s}{N_p}
\end{aligned}$$

(where N_p, N_s are the number of turns on the primary and secondary coils respectively) because these equations assume very good coupling between the coils ($L_1 L_2 - M^2 \approx 0$). Not only is the coupling nowhere near this good, these equations yield a first order system that does not exhibit a resonance. L_2 can be treated as a dependent voltage source. What this tells us is that extremely good coupling *kills* the resonance.

The real way to model the system is (this is a good model. the only things it doesn't model in the electrostatic limit are parasitics):

Transformer Equations.

$$V_1 = L_1{I_1}' + MI_2'$$

$$V_2 = L_2I_2' + MI_1'$$

$$V_1 = j\omega L_1I_1 + j\omega MI_2$$

$$V_2 = j\omega L_2I_2 + j\omega MI_1$$

Basic Relations

$$V_C = \frac{q}{C}$$

$$I_2 = q'$$

Calculations

$$V_C = -V_R - V_2$$

$$V_C = -I_2R - (L_2I_2' + MI_1')$$

$$V_C = -I_2R - (L_2I_2' + MI_1')$$

$$V_C = -I_2R - (L_2I_2' + \frac{M}{L_1}(V - MI_2'))$$

Rearranging the equation yields

$$(L_1L_2 - M^2)q'' + L_1Rq' + \frac{L_1}{C}q = -MV$$

Looking at this equations confirms that if $L_1L_2 - M^2 \approx 0$ then the system is in fact first order as stated above. A general theorem about coupled inductors states that $L_1L_2 - M^2 > 0$ (reitz, p. 242).

Put it in standard form:

$$q'' + \frac{RL_1}{L_1L_2 - M^2}q' + \frac{L_1}{C(L_1L_2 - M^2)}q = \frac{-M}{L_1L_2 - M^2}V$$

$$2\alpha = \frac{RL_1}{L_1L_2 - M^2}$$

$${\omega_0}^2 = \frac{L_1}{C(L_1L_2 - M^2)}$$

The transfer function from V to q looks just like a series RLC circuit with $R =?$ drive by V.

4.1.1 Power

The above power development is inadequate for this circuit because q' is *not* the current moving through the voltage source. We will use impedence methods. The average power P dissipated in this circuit is

$$P = \frac{1}{2} Re\{\overline{I}V\}$$

Using the above transformer equations, it is simple to show that

$$V_1 = (j\omega L_1 + \frac{\omega^2 M^2}{R + \frac{1}{j\omega C} + j\omega L_2})I_1$$

That is, the impedence of the circuit is

$$Z = j\omega L_1 + \frac{\omega^2 M^2}{R + \frac{1}{j\omega C} + j\omega L_2}$$

If we drive the circuit with a current source $I_0 e^{j\omega t}$,

$$\begin{aligned} P &= \frac{(I_0 \omega M)^2}{2} \frac{1}{\sqrt{(\omega L_2 - \frac{1}{\omega C})^2 + R^2}} \cos\left(\arctan \frac{\omega L_2 - \frac{1}{\omega C}}{R}\right) && (2) \\ &= \frac{(I_0 \omega M)^2}{2} \frac{R}{(\omega L_2 - \frac{1}{\omega C})^2 + R^2} && (3) \end{aligned}$$

The maximum average power dissipated, P_m occurs at $\omega = \omega_0$:

$$P_m = \frac{(I_0 \omega_0 M)^2}{2R}$$

Question: What value of the coupling coefficient gives you optimal power transfer to the tag?

4.2 Capacitive Coupling

4.2.1 Tag without the body without parasitic electrode capacitances

$$q'' + \frac{R}{L} q' + \frac{1}{LC} q = \frac{V_0}{L} e^{j\omega t}$$

where $C = \frac{C_1 C_2}{C_1 + C_2}$.

$$\alpha = \frac{R}{2L}$$

$${\omega_0}^2 = \frac{1}{LC}$$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$P_m = \frac{{V_0}^2}{2R}$$

4.2.2 Tag without body with interelectrode capacitances

If we model the parasitic capacitances between electrodes then the systems turns into a parallel RLC tank.

4.2.3 Tag with body without interelectrode capacitances where the tag is just a series L and R

We can ignore interelectrode capacitances due to the distance between electrodes.

$$\frac{V}{I} = Z = H(s) = \frac{s^3 L(1+\frac{C_4}{C_3}) + s^2 R(1+\frac{C_4}{C_3}) + s\frac{C_3+C+C_4}{CC_3}}{s^2 L + sR + \frac{C+C_3}{C_4 C_3 C}}$$

$$C = \frac{C_1 C_2 C_s}{C_1 C_2 + C_1 C_s + C_2 C_s}$$

5 notes

look through microwave solid state circuit design resonator section. this book is in the lab.

multiple definitions of q=average stored energy/energy dissipated per radian = center frequency/half power points

voltage gain = q * v

purely resistive at resonance

do series resonance example

do everything in terms of Q

field of antenna, relation to electrostatic limit

energy in resonator

do loaded Q relationships

antiresonance by parasitic capacitances

6 references

A.P. French, Vibration and Waves, 8.03 tex

6.002 text